

Multi-Lagrangians for Integrable Systems

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Abstract

We propose a general scheme to construct multiple Lagrangians for completely integrable non-linear evolution equations that admit multi-Hamiltonian structure. The recursion operator plays a fundamental role in this construction. We use a conserved quantity higher/lower than the Hamiltonian in the potential part of the new Lagrangian and determine the corresponding kinetic terms by generating the appropriate momentum map.

This leads to some remarkable new developments. We show that non-linear evolutionary systems that admit N -fold first order local Hamiltonian structure can be cast into variational form with $2N - 1$ Lagrangians which will be local functionals of Clebsch potentials. Furthermore we construct a new Lagrangian for polytropic gas dynamics in $1 + 1$ dimensions which is a *local* functional of the physical field variables, namely density and velocity, thus dispensing with the necessity of introducing Clebsch potentials entirely. This is a consequence of bi-Hamiltonian structure with a compatible pair of first and third order Hamiltonian operators derived from Sheftel's recursion operator. There are in fact infinitely many such local Lagrangians.

1 Introduction

In this paper we shall point out a general technique for the construction of inequivalent solutions to the inverse problem in the calculus of variations. We shall show that completely integrable partial differential equations in $1 + 1$ dimensions that admit multi-Hamiltonian structure can be cast into variational form with multiple Lagrangians. It is remarkable that all these new Lagrangians can be obtained directly from our present knowledge of complete integrability of the evolutionary system without doing any new calculations!

One of the important properties we expect from a completely integrable system is multi-Hamiltonian structure. A vector evolutionary system can

then be cast into Hamiltonian form in more than one way

$$u_{t_{\aleph+\alpha-1}}^i = \{u^i, H_\alpha\}_\aleph = J_\aleph^{ik} \delta_k H_\alpha \quad \left\{ \begin{array}{l} i = 1, 2, \dots, n \\ \aleph = 1, 2, \dots, N. \\ \alpha = -1, 0, 1, \dots, \infty \end{array} \right. \quad (1)$$

where the variational derivative is denoted by $\delta_k \equiv \delta/\delta u^k$ and J is a matrix of differential operators satisfying the properties of a Poisson tensor, namely skew-symmetry and Jacobi identity. For integrable systems there exists more than one such Hamiltonian operator and Hamiltonian function as the respective Hebrew and Greek indices indicate. Then, by the theorem of Magri [1] completely integrable systems admit infinitely many conserved Hamiltonian functions which are in involution with respect to Poisson brackets defined by compatible Hamiltonian operators.

The essential element in the multi-Hamiltonian approach to integrability is the construction of the Hamiltonian operators themselves. Fortunately this is a rich subject [2] that can be put to good use. We shall be interested in the consequences of multi-Hamiltonian structure on the Lagrangian formulation of completely integrable evolutionary equations. We shall work in the opposite direction to the traditional approach of deriving Hamiltonian structure from a Lagrangian. The crucial fact that we shall exploit is the relationship between Hamiltonian operators and Dirac brackets [3] for degenerate Lagrangian systems which was first pointed out by Macfarlane [4]. In the case of completely integrable systems we have much more information on Hamiltonian structure than Lagrangian and it became clear only recently [5], [6], [7] how we can construct multiple Lagrangians for systems that admit multi-Hamiltonian structure. We shall now present the general and most simple technique for generating these new Lagrangians.

2 Multi-Lagrangians

Evolutionary systems (1) cannot be cast into variational form with a local expression for the Lagrangian using the velocity fields u^i alone but require the introduction of Clebsch potentials. In 1+1-dimensions the general expression for Clebsch potentials is given by

$$u^i = \phi_x^i \quad (2)$$

and in this paper we shall only consider Lagrangians that are local functionals of these potentials. In the time-honored way we shall split the Lagrangian density for eqs.(1) into two

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \quad (3)$$

that consist of the kinetic and potential pieces respectively. For the first Lagrangian density, an enumeration which will become clear presently, the kinetic term is always given by

$$\mathcal{T}_1 = g_{ik} \phi_t^i \phi_x^k \quad (4)$$

where g_{ik} are constants with $\det g_{ik} \neq 0$ and

$$\mathcal{V}_1 = 2\mathcal{H}_1 \quad (5)$$

is the Hamiltonian density. We note that the Hamiltonian function that appears in (1) is the space integral of the density. We shall number the conserved Hamiltonians by reserving the subscript 1 to the “usual” Hamiltonian function but of course there exists conserved quantities such as Casimirs and the momentum which are of lower order. In fact, for complete integrability, an n -component vector evolutionary system (1) must admit n infinite series of conserved Hamiltonians. We shall denote their densities by

$$\mathcal{H}_{\alpha;[i]} \quad i = 1, 2, \dots, n; \quad \alpha = -1, \dots, \infty \quad (6)$$

and recall that each series starts with a Casimir

$$\mathcal{H}_{-1;[i]} = g_{ik} u^k \quad (7)$$

which will carry the label minus one. One of these series is distinguished in that it contains the “usual” Hamiltonian function which is the one that appears in eq.(1). For the 2-component systems that we shall discuss in this paper these are the Eulerian and Lagrangian series. We note also that the two series may coincide up to a relabelling dictated by the recursion operator. This is in fact the case for the $\gamma = 2$ case of gas dynamics and in most examples of completely integrable dispersive equations except the Boussinesq equation.

The potential part of the Lagrangian does not depend on the velocities and from eq.(4) it follows that the Hessian

$$\det \left| \frac{\partial^2 \mathcal{L}_1}{\partial \phi_t^i \partial \phi_t^k} \right| = 0$$

vanishes identically. We have therefore a degenerate Lagrangian system and in order to cast it into Hamiltonian form we must use Dirac's theory of constraints [3], or the covariant Witten-Zuckerman theory [8, 9] of symplectic structure. In particular, the first Hamiltonian operator obtained from the first Lagrangian is given by

$$J_1^{ik} = g^{ik} D \quad D \equiv \frac{d}{dx} \quad (8)$$

where g^{ik} is the inverse of the coefficients in the kinetic part of the first Lagrangian (4) which is non-degenerate.

The construction of multiple Lagrangians relies on the use of the Lenard recursion relation which is implicit in eqs.(1) that in the Greek and Hebrew indices we have a symmetric matrix

$$J_{[\aleph]k}^{ik} \delta_{|k|} H_{\alpha]} = 0 \quad (9)$$

where square brackets denote complete skew-symmetrization and bars enclose indices which are excluded in this process. Provided we can invert these Hamiltonian operators, we can construct recursion operators

$$R_{\aleph_2}^{\aleph_1}{}^i{}_k = J_{\aleph_2}^{im} (J_{\aleph_1}^{mk})^{-1} \quad (10)$$

that map gradients of conserved Hamiltonians into each other (9).

For the construction of Lagrangians we start with the crucial observation that the first Lagrangian is of the form

$$\mathcal{L}_1 = \mathcal{H}_{-1[i]} \phi_t^i - 2\mathcal{H}_1 \quad (11)$$

which is manifest from (4). The original fields that enter into the evolutionary system (1) are Casimirs which is evident from the subscript minus one. The second Lagrangian will be of the same general structure as (11) if we further suppose that eqs.(1) can be written in bi-Hamiltonian form. Thus there will exist H_2 which is the next conserved Hamiltonian function in the hierarchy and the momentum H_0 which comes after Casimirs. The higher Lagrangian should simply be

$$\mathcal{L}_2 = \mathcal{H}_{0[i]} \phi_t^i - 2\mathcal{H}_2$$

but there is an important refinement that we need to insert here. It is not the conserved density but rather the momentum map that enters into

the kinetic part of the Lagrangian. The two differ only by total derivatives which is irrelevant in the context of conservation laws and therefore generally skipped over. However, these divergence terms are of crucial interest as the momentum map in the theory of symplectic structure. We shall show that given α^{th} local Hamiltonian structure, the full new Lagrangian is simply given by

$$\mathcal{L}_\alpha = \{\mathcal{H}_{\alpha-2[i]} + (\mathcal{G}_{\alpha-2[i]})_x\}\phi_t^i - 2\mathcal{H}_\alpha \quad (12)$$

where $\mathcal{G}_{\alpha[i]}$ is a functional of the potentials. The coefficient of ϕ_t^i above is the momentum map and this is the only calculation necessary to find the new Lagrangian.

The fact that it is the momentum map rather than the conserved density that plays an important role in the Lagrangian can be seen at the level of the first Lagrangian. Now the Casimirs play the role of the momentum map and they are used to construct the next higher conserved quantity according to the construction of the canonical energy-momentum tensor

$$\mathcal{H}_0 = \frac{\partial \mathcal{L}_1}{\partial \phi_t^i} \phi_x^i = \mathcal{H}_{-1[i]} u^i = \frac{1}{2} g^{ik} \mathcal{H}_{-1[i]} \mathcal{H}_{-1[k]}$$

which is the momentum. This classical result for Lagrangians linear in the velocity can be generalized at each level we have a higher Lagrangian. We have

$$2\mathcal{H}_{\alpha-1} = g^{ik} [\mathcal{H}_{\alpha-2[i]} + (\mathcal{G}_{\alpha-2[i]})_x] \mathcal{H}_{-1[k]} \quad (13)$$

ending at the level where a local Lagrangian is no longer possible. In fact the validity of this equation is directly related to the existence of the Lagrangian. If a check of (13) fails for some α , then there exists no local Lagrangian at α^{th} level.

Now we come to an important reservation that our new Lagrangians will necessarily carry. The Euler equations that follow from the variation of the action with the second Lagrangian will be

$$R_{2\ k}^{1\ i} \left[u_t^k - J_1^{km} \delta_m H_1 \right] = 0 \quad (14)$$

so that the first variation of the second action will certainly be an extremum for the original equations of motion (1) but the Euler equations (14) require something weaker, namely linear combinations of functionals in the kernel of the recursion operator can be added to the right hand side of the equations of motion and the new action will still be an extremum.

From this construction it is manifest that for every Hamiltonian function in the infinite hierarchy of conserved Hamiltonians that we have for completely integrable systems, there exists a degenerate Lagrangian (12) that yields the equations of motion as its Euler equation up to functionals in the kernel of the recursion operator. The number of Lagrangians that can be constructed in this way is therefore infinite in number. Given bi-Hamiltonian structure we have two local Hamiltonian operators but the Lenard recursion operator (10) is non-local. However, the special form of the first Hamiltonian operator (8) leads to a local expression for the second Lagrangian in terms of Clebsch potentials. But it is clear that the repeated application of the recursion operator will require the introduction of non-local terms in higher Lagrangians. Strictly speaking, this is not a problem because the original Lagrangian is itself non-local in terms of the velocity fields u^i which are the original variables. We swept this problem under the rug by introducing Clebsch potentials. Higher Lagrangians for evolutionary equations (1) can be written in local form by introducing potentials for the Clebsch potentials themselves! If, however, the equations of motion admit n local Hamiltonian operators, then our construction guarantees the existence of n Lagrangians which are local functionals of the Clebsch potentials. Thus we have

Theorem 1 *A completely integrable system that admits n -fold local first order Hamiltonian structure can be given n different variational formulations with degenerate Lagrangians that are local functionals of the Clebsch potentials.*

By a convenient abuse of language we claim that we have a Lagrangian for an equation that involves fields when the Lagrangian is in fact only a functional of the Clebsch potentials for these fields. Then we have the audacity to put in by hand the expression for the fields in terms of potentials after the variation! This can be at best only a shorthand for the real variational principle where we must impose the relationship between the fields and their potentials through Lagrange multipliers.

So far we have been guilty of this abuse ourselves. But now we must say that the first Lagrangian is actually

$$\mathcal{L}_{\geq 1}^{full} = \mathcal{L}_1(\phi^i, \phi_x^i, \phi_{xx}^i \dots) + \lambda_i(u^i - \phi_x^i) \quad (15)$$

so that upon variation with respect to all the variables ϕ^i, u^i, λ_i we get (2), $\lambda_i = 0$ and we arrive at the equations of motion (1) expressed in terms of the original fields u^i without fudging.

Now this obvious observation may seem correct but naive, however, we shall now find that it dramatically increases the number of new Lagrangians we can construct for integrable systems.

For every evolutionary equation that admits, say for simplicity, bi-Hamiltonian structure there exists a differential substitution

$$u^i = M^i(r^k, r_x^k, \dots) \quad (16)$$

that brings the second Hamiltonian operator to the canonical form (8) of Darboux. This differential substitution is a Miura transformation. Strictly speaking the theorem of Darboux remains unproved in field theory where the number of degrees of freedom is infinite but we shall assume it. Miura transformation works in a direction opposite to the usual action of the recursion operator. It leads to Hamiltonian equations

$$r_t^i = \{r^i, H_0\}_1 = g^{ik} D \delta_{r^k} H_0 \Big|_{u^m = M^m(r^n)} \quad (17)$$

where H_0 is the momentum for eqs.(1) expressed through (16). These are modified equations, different from the original equations, but the two sets are related by

$$u_t^i - J_2^{ik} \delta_{u^k} H_0(u) = \mathcal{O}_j^i(r^l) \left[r_t^j - J_1^{jk} \delta_{r^k} H_0 \Big|_{u^m = M^m(r^n)} \right] \quad (18)$$

up to functions in the kernel of some matrix differential operator \mathcal{O}_j^i .

A comparison of eqs.(14) and Miura's relation (18) shows us that using the differential substitution of Miura we can obtain new Lagrangians for nonlinear evolution equations that admit multi-Hamiltonian structure in the opposite direction to our earlier construction. Transforming to the variables r^i and using Clebsch potentials

$$r^i = \psi_x^i \quad (19)$$

we can write the classical Lagrangian for the modified system (17)

$$\mathcal{L}_1^{modified} = g_{ik} \psi_x^i \psi_t^k - 2\mathcal{H}_0 \Big|_{u^m = M^m(\psi_x, \psi_{xx} \dots)} \quad (20)$$

where the labelling of \mathcal{H}_0 refers to its expression in the original variables u^i but these need to be substituted for in terms of r^i according to (16) and expressed through the potentials (19). We note that the Casimirs $r^i = \psi_x^i$ for

the modified system are absent in the polynomial $\mathcal{H}_\alpha(u)$ hierarchy. We would expect naively that the Euler equations resulting from the first variation of the action with the Lagrangian (20) would result in the modified equations (17). This would indeed be the case if we were to impose the constraint between the fields r^i and their potentials ψ^i as in (15) but now using (19). However, by imposing the constraint through Miura's differential substitution

$$\mathcal{L}_0^{full} = \mathcal{L}_1^{modified}(\psi_x^i, \psi_{xx}^i \dots) + \lambda_i[u^i - M^i(\psi_x^i, \psi_{xx}^i \dots)] \quad (21)$$

we obtain a new Lagrangian for the original equations (1) in the original variables u^i . We shall use this construction to derive new Lagrangians, in particular for KdV in section 3. It is evident that this construction can be extended when there exists multi-Hamiltonian structure but, as we shall find in the example of KdV, sometimes it is possible to arrive at local Lagrangians using non-local Hamiltonian operators as well. Now we conclude

Theorem 2 *The first Lagrangian of every modified equation obtained through a Miura transformation will serve as a new zeroth Lagrangian for the original equations of motion provided the constraint between the fields and their potentials is imposed through a Miura-type differential substitution. For N -fold Hamiltonian structure there exists $N - 1$ such new Lagrangians.*

Miura transformation is in general not invertible because it is a differential substitution. But there exists interesting examples where it reduces to a point transformation which is invertible. In that case we can construct $N - 1$ further Lagrangians.

We conclude that for an evolutionary system that admits N fold first order Hamiltonian structure, the number of different variational principles where the first variation will be an extremum by virtue of the original equations of motion is $2N - 1$ and in the case Miura transformation is invertible $3N - 2$. We illustrate this situation for the case of bi-Hamiltonian structure in tables 1 and 2. The general situation is much more complicated than what these tables would lead us to expect. Starting with tri-Hamiltonian structure the individual entries in each one of these tables will need to be table by itself because there are inequivalent Hamiltonian operators that yield the same equations of motion with the same Hamiltonian function. We shall discuss this interesting situation in a future publication on the Chaplygin-Born-Infeld equation.

equations of motion	$u_t = J_2 \delta H_0$	$u_t = J_1 \delta H_1$	$J_2 J_1^{-1} u_t = -\delta_\phi H_2$
local Hamiltonian op.	J_2	J_1	no
local Lagrangian	no	\mathcal{L}_1	\mathcal{L}_2
modified equations	$r_t = \tilde{J}_2 \delta H_0$	$r_t = \tilde{J}_1 \delta H_1$	
local Hamiltonian op.	$\tilde{J}_2 = J_1$	\tilde{J}_1	
local Lagrangian	\mathcal{L}_0	no	

Table 1: The hierarchy of local Hamiltonian structures and Lagrangians which are local functionals of Clebsch potentials for evolutionary system $u_t = J_1 \delta H_1 = J_2 \delta H_0$ where J_1 is in the canonical Darboux form.

equations of motion		$u_t = J_2 \delta H_0$...
local Hamiltonian op.		J_2	...
local Lagrangian		no	...
modified equations	$\tilde{J}_1 \tilde{J}_2^{-1} r_t = -\delta_\psi H_{-1}$	$r_t = \tilde{J}_2 \delta H_0$...
local Hamiltonian op.	no	$\tilde{J}_2 = J_1$...
local Lagrangian	\mathcal{L}_{-1}	\mathcal{L}_0	...

Table 2: When the Miura transformation is invertible we need to include an additional column to the left of table 1.

3 KdV

KdV stands as the symbol of completely integrable systems. We think we know it, but it turns out to be so rich that there is still new information to be learned about it. We recall that KdV

$$u_t + 6u u_x - u_{xxx} = 0 \quad (22)$$

admits the Kruskal sequence of conserved Hamiltonian densities

$$\mathcal{H}_{-1}^{KdV} = u \quad (23)$$

$$\mathcal{H}_0^{KdV} = \frac{1}{2}u^2 \quad (24)$$

$$\mathcal{H}_1^{KdV} = u^3 + \frac{1}{2}u_x^2 \quad (25)$$

$$\mathcal{H}_2^{KdV} = \frac{5}{2}u^4 + 5u u_x^2 + \frac{1}{2}u_{xx}^2 \quad (26)$$

...

which are in involution with respect to Poisson brackets defined by two Hamiltonian operators

$$J_1 = D, \quad J_2 = -D^3 + 2uD + 2Du \quad (27)$$

that form a Poisson pencil. By introducing the potential

$$u = \phi_x \quad (28)$$

KdV can be cast into variational form with two Lagrangians

$$\mathcal{L}_1^{KdV} = \mathcal{H}_{-1}^{KdV} \phi_t - 2\mathcal{H}_1^{KdV} \quad (29)$$

$$\mathcal{L}_2^{KdV} = (\mathcal{H}_0^{KdV} + \phi_{xx})\phi_t - 2\mathcal{H}_2^{KdV} \quad (30)$$

which consist of the classical Lagrangian and the second Lagrangian [5] respectively. The detailed derivation of (30), which in [6] was called Pavlov's Lagrangian, exists in that reference. Here we shall simply note that both (29) and (30) are examples of our general expression (12) for higher Lagrangians.

The second application of Lenard's recursion operator to J_1 results in a third Hamiltonian operator which is non-local so we cannot continue to generate higher Lagrangians. But we can use Theorem 2 to generate new lower

Lagrangians for KdV. For this purpose we note that in both Lagrangians (29) and (30) we should have added the constraint $\lambda(u - \phi_x)$ and written the full Lagrangian. But following the convenient abuse of language we did not do so because it was manifest. It is, however, necessary to write the full Lagrangian in the case of lower Lagrangians.

According to our general construction of lower Lagrangians we first recall the original Miura transformation

$$u = r^2 + r_x \quad (31)$$

that brings J_2 to the canonical form of J_1 in the variable r . The equation of motion for r is mKdV which is different from (22) but under the substitution (31) we have Miura's result

$$u_t + 6u u_x - u_{xxx} = (D + 2r) (r_t + 6r^2 r_x - r_{xxx}) = 0 \quad (32)$$

so that, on shell, if mKdV is satisfied then so is KdV. Now we can introduce the Clebsch potential for the modified field variable

$$r = \psi_x \quad (33)$$

and write the first Lagrangian for mKdV

$$\mathcal{L}_1^{mKdV} = \psi_x \psi_t + \mathcal{H}_0^{KdV} \Big|_{u=\psi_x^2+\psi_{xx}} \quad (34)$$

in a straight-forward manner. But now enforcing the constraint in the full Lagrangian through the Miura transformation

$$\mathcal{L}_0^{KdV \text{ full}} = \mathcal{L}_1^{mKdV} + \lambda(u - \psi_x^2 - \psi_{xx}) \quad (35)$$

we shall arrive at a new Lagrangian for KdV because the Euler equation that comes from the first variation of this action will be satisfied by virtue of (32). Unlike (30) which is a higher Lagrangian, (35) is a lower Lagrangian in the sense of the action of the recursion operator on the equations of motion in the resulting Euler equation.

And the saga of KdV continues! We consider the third Hamiltonian operator for KdV

$$J_3 = R^2 J_1 \quad (36)$$

which is nonlocal but the relationship between differential substitutions and Hamiltonian structures of KdV [23] enables us to construct another new

local Lagrangian for KdV. For this purpose we recall that the differential substitution

$$r = \alpha q + \frac{\varepsilon}{q} + \frac{q_x}{2q} \quad (37)$$

which transforms mKdV into twice modified KdV

$$q_t = \left(q_{xx} - \frac{3q_x^2}{2q} + \frac{6\varepsilon^2}{q} - 2\alpha^2 q^3 \right)_x \quad (38)$$

is a Miura transformation for (36). This can best be seen by the expression

$$J_3 = \frac{1}{2}(q^2 D + D q^2) - q_x D^{-1} q_x \quad (39)$$

for the third non-local Hamiltonian operator for KdV in terms of twice modified variable q . We recall that J_3 is fifth order in u . We have the Miura relation

$$\begin{aligned} r_t + 6r^2 r_x - r_{xxx} &= \left(\alpha - \frac{\varepsilon}{q^2} - \frac{q_x}{2q^2} + \frac{1}{q} D \right) \\ &\quad \left[q_t - \left(q_{xx} - \frac{3q_x^2}{2q} + \frac{6\varepsilon^2}{q} - 2\alpha^2 q^3 \right)_x \right] \\ &= 0 \end{aligned} \quad (40)$$

between modified and twice modified KdV's. Introducing the potential for the twice modified variable $q = \chi_x$ we have

$$u = \Phi(\chi_x, \chi_{xx}, \chi_{xxx}) \equiv \frac{\chi_{xxx}}{\chi_x} - \frac{\chi_{xx}^2}{\chi_x^2} + 2\alpha\chi_{xx} + \alpha^2\chi_x^2 + 2\alpha\varepsilon + \frac{\varepsilon^2}{\chi_x^2} \quad (41)$$

in terms of the original field u . The first Lagrangian for twice modified KdV is simply

$$\mathcal{L}_1^{m_2 KdV} = \chi_x \chi_t + \mathcal{H}_{-1}^{KdV} \Big|_{u=\Phi(\chi_x, \chi_{xx}, \chi_{xxx})} \quad (42)$$

and therefore the second lower Lagrangian for KdV is given by

$$\mathcal{L}_{-1}^{KdV \text{ full}} = \mathcal{L}_1^{m_2 KdV} + \lambda[u - \Phi(\chi_x, \chi_{xx}, \chi_{xxx})] \quad (43)$$

which provides another illustration of (21). This process can be continued.

We note that an alternative to the Clebsch potential for KdV is the Schwartzian which was pointed out by Schiff [10]. We shall postpone consideration of Schwartzian potentials to future work.

4 Polytropic gas dynamics

The simplest examples for applying our construction of multi-Lagrangians consist of quasi-linear second order hyperbolic equations that Dubrovin and Novikov [11] have called equations of hydrodynamic type. The distinguished example in this set consists of the Eulerian equations of polytropic gas dynamics in 1 + 1 dimensions

$$\begin{aligned}\rho_t + u \rho_x + \rho u_x &= 0 \\ u_t + u u_x + \rho^{\gamma-2} \rho_x &= 0\end{aligned}\tag{44}$$

and in particular for $\gamma = -1$ we have the case of Chaplygin gas, or Born-Infeld equation that was recently shown to have a string theory antecedent [12]. This system can be cast into quadri-Hamiltonian form [13]. For the Chaplygin-Born-Infeld case see [14]. In the following we shall use the labelling $u^1 = \rho$ and $u^2 = u$.

First we have three local Hamiltonian structures of first order [15]

$$J_1 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} = \sigma^1 D, \tag{45}$$

$$J_2 = \begin{pmatrix} \rho D + D \rho & (\gamma - 2) D u + u D \\ D u + (\gamma - 2) u D & \rho^{\gamma-2} D + D \rho^{\gamma-2} \end{pmatrix}, \tag{46}$$

$$J_3 = \begin{pmatrix} u \rho D + D u \rho & D \left[\frac{1}{2}(\gamma - 2)u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} \right] \\ D \left[\frac{1}{2}u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} \right] & + \left[\frac{1}{2}u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} \right] D \\ + \left[\frac{1}{2}(\gamma - 2)u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} \right] D & u \rho^{\gamma-2} D + D u \rho^{\gamma-2} \end{pmatrix} \tag{47}$$

which form a Poisson pencil $\mathcal{J} = J_1 + c_1 J_2 + c_2 J_3$ with c_1, c_2 constants, *i.e.* these Hamiltonian operators are compatible. In eq.(45) σ^1 is the Pauli matrix and this is the canonical Darboux form of first order Hamiltonian operators. The equations of polytropic gas dynamics admit two infinite hierarchies of conserved Hamiltonians which are in involution with respect to Poisson brackets defined by all three of these Hamiltonian operators. In the first set, which is called Eulerian [13], the Hamiltonian densities are given by

$$\mathcal{H}_{-1}^E = \rho \tag{48}$$

$$\mathcal{H}_0^E = u \rho \quad (49)$$

$$\mathcal{H}_1^E = \frac{1}{2}u^2\rho + \frac{1}{\gamma(\gamma-1)}\rho^\gamma \quad (50)$$

$$\mathcal{H}_2^E = \frac{1}{6}u^3\rho + \frac{1}{\gamma(\gamma-1)}u\rho^\gamma \quad (51)$$

$$\mathcal{H}_3^E = \frac{1}{24}u^4\rho + \frac{1}{2\gamma(\gamma-1)}u^2\rho^\gamma + \frac{1}{2\gamma(\gamma-1)^2(2\gamma-1)}\rho^{2\gamma-1} \quad (52)$$

...

where (49) is the momentum, (50) is the familiar Hamiltonian function, the Casimir is in (48) and the rest consist of higher Hamiltonians. Therefore, the Euler series is the distinguished one in the terminology of section 2. The second series

$$\mathcal{H}_{-1}^L = u \quad (53)$$

$$\mathcal{H}_0^L = \frac{1}{2}(\gamma-2)u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1} \quad (54)$$

$$\mathcal{H}_1^L = \frac{1}{6}(\gamma-2)u^3 + \frac{1}{\gamma-1}u\rho^{\gamma-1} \quad (55)$$

$$\mathcal{H}_2^L = \frac{1}{24}(\gamma-2)u^4 + \frac{1}{4(\gamma-1)}u^2\rho^{\gamma-1} + \frac{1}{2(\gamma-1)^2(2\gamma-3)}\rho^{2(\gamma-1)} \quad (56)$$

...

is the Lagrangian series which starts with the Casimir (53). Note that for $\gamma = 2$ this series is no longer polynomial as logarithms will enter and the same remark holds for integer and half-integer values of γ in both series.

Finally, we note that the recursion operator $R_2^1 = J_2 J_1^{-1}$ can be used to write infinitely many Hamiltonian operators by letting it to act n times on J_1 . However, in general none of these operators will be local. In particular we note that

$$R_3^1 = J_3 (J_1)^{-1} \neq (R_2^1)^2, \quad J_3 \neq J_2 J_1^{-1} J_2 \quad (57)$$

except in the case of shallow water waves where $\gamma = 2$ which admits extension to integrable dispersive equations.

Next, there is a third order Hamiltonian operator [16] which was obtained from Sheftel's remarkable recursion operator [17]

$$J_4 = D U_x^{-1} D U_x^{-1} \sigma^1 D \quad (58)$$

where

$$U = \begin{pmatrix} u & \rho \\ \frac{1}{\gamma-2}\rho^{\gamma-2} & u \end{pmatrix} \quad (59)$$

which is only compatible with J_0 . Higher conserved Hamiltonians start with the density [17], [18]

$$\hat{\mathcal{H}}_{-1}^{SV(E)} = \frac{\rho_x}{u_x^2 - \rho^{\gamma-3}\rho_x^2} \quad (60)$$

which is part of the Eulerian series. There is also a Lagrangian series starting with

$$\hat{\mathcal{H}}_{-1}^{SV(L)} = -\frac{u_x}{u_x^2 - \rho^{\gamma-3}\rho_x^2} \quad (61)$$

and both form new infinite hierarchies of conservation laws.

We will be interested in the Lagrangian formulation of the equations of polytropic gas dynamics (44) that correspond to all these Hamiltonian structures. Introducing the Clebsch potentials [19]

$$u = \varphi_x, \quad \rho = \psi_x \quad (62)$$

we have the first Lagrangian representation for this system

$$\mathcal{L}_1^\gamma = \mathcal{H}_{-1}^L \psi_t + \mathcal{H}_{-1}^E \varphi_t - 2\mathcal{H}_1^E(\varphi_x, \psi_x) \quad (63)$$

but using the recursion operators $J_2 J_1^{-1}$ and $J_3 J_1^{-1}$ we find two further Lagrangians

$$\mathcal{L}_2^\gamma = \mathcal{H}_0^L \psi_t + \mathcal{H}_0^E \varphi_t - 2\mathcal{H}_2^E(\varphi_x, \psi_x) \quad (64)$$

$$\mathcal{L}_3^\gamma = \mathcal{H}_1^L \psi_t + \mathcal{H}_1^E \varphi_t - 2\mathcal{H}_3^E(\varphi_x, \psi_x) \quad (65)$$

which are local functionals of the Clebsch potentials. The Lagrangian obtained through the action of the recursion operator $J_4 J_1^{-1}$ is the most interesting one. Because J_4 is a third order operator, the fourth Lagrangian

$$\mathcal{L}_4^\gamma = \mathcal{H}_{-1}^{SV(E)} u_t + \mathcal{H}_{-1}^{SV(L)} \rho_t - 2\mathcal{H}_{-1}^E(\varphi_x, \psi_x) \quad (66)$$

$$\mathcal{L}_4^\gamma = \frac{\rho_x u_t - u_x \rho_t}{u_x^2 - \rho^{\gamma-3}\rho_x^2} - 2\rho \quad (67)$$

is *local in the velocity fields*. This is a general property of bi-Hamiltonian structure with a pair of first and third order Hamiltonian operators. Here we find a remarkable situation in that the number of Lagrangians that we

can construct by repeated application of Sheftel's recursion operator $J_4 J_1^{-1}$ is *infinite* in number. All of these Lagrangians will be *local* in the original field variables ρ and u .

Now we come to lower Lagrangians that will arise from Miura transformations. The Miura transformations that bring the Hamiltonian operators (46) and (47) to the Darboux form of (45) are point transformations for equations of hydrodynamic type. Dubrovin and Novikov had pointed out that first order Hamiltonian operators for equations of hydrodynamic type are given by

$$J^{ik} = g^{ik} D - g^{im} \Gamma_{mn}^k u_x^n \quad (68)$$

where g_{ik} are the components of a Riemannian metric which is flat by virtue of the Jacobi identities. The Miura transformation provides manifestly flat coordinates for this metric. For example from (46) we find the flat metric

$$ds_2^2 = \frac{2}{4\rho^{\gamma-1} - (\gamma-1)^2 u^2} \left[\rho^{\gamma-2} d\rho^2 - (\gamma-1)u d\rho du + \rho du^2 \right] \quad (69)$$

and it can be verified that the Miura transformation

$$\rho = r p \quad u = \frac{1}{\gamma-1} (r^{\gamma-1} + p^{\gamma-1}) \quad (70)$$

brings it into the manifestly flat form $2dr dp$. In these variables we find the first modified equations of gas dynamics

$$\begin{aligned} r_t + \frac{\gamma}{\gamma-1} (r^{\gamma-1} + p^{\gamma-1}) r_x + \gamma r p^{\gamma-2} p_x &= 0 \\ p_t + \gamma p r^{\gamma-2} r_x + \frac{\gamma}{\gamma-1} (r^{\gamma-1} + p^{\gamma-1}) p_x &= 0 \end{aligned} \quad (71)$$

and linear combinations of these equations with variable coefficients give eqs.(44) of gas dynamics. Introducing the potentials

$$r = \chi_x, \quad p = v_x \quad (72)$$

we have the Lagrangian

$$\mathcal{L}_0^{\gamma full} = \chi_x v_t + v_x \chi_t - 2\mathcal{H}_0^E + \lambda \left(u - \frac{\chi_x^{\gamma-1} + v_x^{\gamma-1}}{\gamma-1} \right) + \sigma (\rho - \chi_x v_x) \quad (73)$$

where \mathcal{H}_0^E is the momentum (49) expressed in terms of the potentials χ and v . Transforming to the first modified variables r, p we get $\tilde{J}_1, \tilde{J}_2 = J_1$ and \tilde{J}_3

defining the tri-Hamiltonian structure of eqs.(71). Now there is a new lower Lagrangian that we can construct from the recursion operator $\tilde{J}_1 \tilde{J}_2^{-1}$. We find

$$\tilde{J}_1 = \begin{pmatrix} (1-\gamma) [rp^{\gamma-2}\Delta D + Drp^{\gamma-2}\Delta] & [(\gamma-2)r^{\gamma-1} + p^{\gamma-1}] \Delta D \\ & + D [r^{\gamma-1} + (\gamma-2)p^{\gamma-1}] \Delta \\ [r^{\gamma-1} + (\gamma-2)p^{\gamma-1}] \Delta D & (1-\gamma) [pr^{\gamma-2}\Delta D + Dpr^{\gamma-2}\Delta] \\ + D [(\gamma-2)r^{\gamma-1} + p^{\gamma-1}] \Delta & \end{pmatrix}, \quad (74)$$

$$\Delta \equiv \frac{1}{(\gamma-1)(r^{\gamma-1} - p^{\gamma-1})^2}$$

where the labelling of the variables is in the order r and p . The new Lagrangian is given by

$$\mathcal{L}_{-1}^{\gamma full} = \frac{\chi_x v_t - v_x \chi_t}{\chi_x^{\gamma-1} - v_x^{\gamma-1}} - \mathcal{H}_{-1}^E + \lambda \left(u - \frac{\chi_x^{\gamma-1} + v_x^{\gamma-1}}{\gamma-1} \right) + \sigma (\rho - \chi_x v_x) \quad (75)$$

where the momenta do not belong to the polynomial series of conserved Hamiltonians. However, we can identify the lower momenta from this Lagrangian

$$\mathcal{H}_{-2}^{\gamma \pm} = \xi_{\pm}^{\frac{3-\gamma}{\gamma-1}} (\xi_+ \xi_-)^{-1/2},$$

$$\xi^2 + (\gamma-1)u\xi + \rho^{\gamma-1} = 0$$

where \pm refers to Eulerian and Lagrangian series as well as the roots of the quadratic equation.

We now turn to the third Hamiltonian structure (47) defined by the flat metric

$$\begin{aligned} ds_3^2 &= -\frac{8(\gamma-1)^2}{[(\gamma-1)^2 u^2 - 4\rho^{\gamma-1}]^2} \{ u\rho^{\gamma-2} d\rho^2 \\ &\quad - \frac{1}{2(\gamma-1)} [(\gamma-1)^2 u^2 + 4\rho^{\gamma-1}] d\rho du + u\rho du^2 \} \\ &= 2dq dw \end{aligned} \quad (76)$$

and the coordinate transformation that brings it to the manifestly flat form is given by

$$q = \left[(\gamma-1)^2 u^2 - 4\rho^{\gamma-1} \right]^{\frac{\gamma-3}{2(1-\gamma)}} \quad (77)$$

$$\begin{aligned}
w &= \int^z \frac{1}{\sqrt{1+\xi^2}} \xi^{\frac{\gamma-3}{1-\gamma}} d\xi \\
z &= \sinh \left\{ \frac{1}{2} \ln \frac{(\gamma-1)u + 2\rho^{(\gamma-1)/2}}{(\gamma-1)u - 2\rho^{(\gamma-1)/2}} \right\}
\end{aligned} \tag{78}$$

where, in general, the last integral cannot be done in closed form. For some specific values of γ the integral (78) is elementary as in the notable case of Chaplygin-Born-Infeld. But this paper is devoted to the general case of polytropic gas dynamics and we shall not consider inverting (77), (78) to obtain u, ρ as functions of q and w . We shall only remark that after this inversion we can obtain two more new Lagrangians.

The Lagrangians (63), (64) and (65) for polytropic gas dynamics are examples illustrating the general expression (12) for higher Lagrangians. For equations of hydrodynamic type there is no dispersion and hence \mathcal{G} vanishes identically. We have given only two (73), (75) of the four lower Lagrangians because the integral (78) must be carried out before we arrive at the second modified equations of gas dynamics which will lead to two further new Lagrangians. Certainly the Lagrangian (67) which is derived from bi-Hamiltonian structure with a first and third order operators according to (12) is the most remarkable one because this is the first time it has been possible to write a Lagrangian for polytropic gas dynamics that is local in the original field variables, namely the density and velocity. Furthermore it is only the first element in an infinite series of such Lagrangians.

5 Kaup-Boussinesq system

Gas dynamics with $\gamma = 2$ governs the behavior of long waves in shallow water. From the point of view of complete integrability it is a remarkable case, because in this case we find several completely integrable dispersive generalizations of eqs.(44). Most prominent among them is the well-known Kaup-Boussinesq system [20]

$$u_t = \left(\frac{u^2}{2} + \rho \right)_x \quad \rho_t = \left(u\rho + \varepsilon^2 u_{xx} \right)_x \tag{79}$$

which admits tri-Hamiltonian structure. The first Hamiltonian structure is given by the Hamiltonian operator (45) and

$$J_2^{KBq} = \begin{pmatrix} D & \frac{1}{2} D u \\ \frac{1}{2} u D & \frac{1}{2} (\rho D + D \rho) + \varepsilon^2 D^3 \end{pmatrix} \quad (80)$$

where D^{-1} denotes the principal value integral, is the second Hamiltonian operator for the Kaup-Boussinesq system. In the limit $\varepsilon \rightarrow 0$ this Hamiltonian operator reduces to (46) with $\gamma = 2$. The recursion operator is given by

$$R_2^{1KBq} = \begin{pmatrix} \frac{1}{2} u + \frac{1}{2} u_x D^{-1} & 1 \\ \varepsilon^2 D^2 + \rho + \frac{1}{2} \rho_x D^{-1} & \frac{1}{2} u \end{pmatrix} \quad (81)$$

and there is a third local Hamiltonian operator obtained by the action of the recursion operator $J_2^{KBq} = (R_2^{1KBq})^2 J_0$ as in the $\gamma = 2$ case of gas dynamics.

The conserved Hamiltonians in the Eulerian and Lagrangian series are

$$\mathcal{H}_{-1}^{KBq} = \rho \quad (82)$$

$$\mathcal{H}_0^{KBq} = u \rho \quad (83)$$

$$\mathcal{H}_1^{KBq} = \frac{1}{2} (\rho u^2 + \rho^2 + \varepsilon^2 u u_{xx}) \quad (84)$$

$$\mathcal{H}_2^{KBq} = \frac{1}{2} [\rho u^3 + 3\rho^2 u - \varepsilon^2 (4u_x \rho_x + 3u u_x^2)] \quad (85)$$

$$\begin{aligned} H_3^{KBq} &= \frac{1}{4} u^4 \rho + \frac{3}{2} u^2 \rho^2 + \frac{1}{2} \rho^3 + \varepsilon^4 u_{xx}^2 \\ &\quad - \varepsilon^2 \left(\frac{5}{2} \rho u_x^2 + 4u u_x \rho_x + \rho_x^2 + \frac{3}{2} u^2 u_x^2 \right) \\ &\quad \dots \end{aligned} \quad (86)$$

and the degeneracy in the $\gamma = 2$ case of gas dynamics is repeated in its dispersive generalization. In particular, the Lagrangian and Eulerian series coincide apart from a relabelling

$$\begin{aligned} \mathcal{H}_{-2}^{KBq(E)} &= u = \mathcal{H}_{-1}^{KBq(L)} \\ \mathcal{H}_{-1}^{KBq(E)} &= \rho = \mathcal{H}_0^{KBq(L)} \\ &\quad \dots \\ \mathcal{H}_{-2+n}^{KBq(E)} &= \mathcal{H}_{-1+n}^{KBq(L)} \end{aligned} \quad (87)$$

that is dictated by the recursion operator.

With the aid of the Clebsch potentials

$$u = \varphi_x, \quad \rho = \psi_x \quad (88)$$

we obtain

$$\mathcal{L}_1^{KBq} = \mathcal{H}_{-1}^{KBq} \varphi_t + \mathcal{H}_{-2}^{KBq} \psi_t - 2\mathcal{H}_1^{KBq}(\varphi_x, \psi_x, \varphi_{xx}, \psi_{xx}, \dots) \quad (89)$$

for the first Lagrangian. Using the technique we have presented in section 2 we shall now construct higher Lagrangians. These three local Hamiltonian structures enable us to construct two new Lagrangians

$$\mathcal{L}_2^{KBq} = (\mathcal{H}_0^{KBq} + \varepsilon^2 \varphi_{xxx}) \varphi_t + \mathcal{H}_{-1}^{KBq} \psi_t - 2\mathcal{H}_2^{KBq}(\varphi_x, \psi_x, \varphi_{xx}, \psi_{xx}, \dots) \quad (90)$$

and

$$\begin{aligned} \mathcal{L}_3^{KBq} = & \left[\mathcal{H}_1^{KBq} + \varepsilon^2 (2\psi_{xxx} + \varphi_{xx}^2 + \varphi_x \varphi_{xxx}) \right] \varphi_t \\ & + (\mathcal{H}_0^{KBq} + \varepsilon^2 \varphi_{xxx}) \psi_t - 2\mathcal{H}_3^{KBq}(\varphi_x, \psi_x, \dots) \end{aligned} \quad (91)$$

for the Kaup-Boussinesq system. The determination of $\mathcal{G}_{\beta;[i]}$ is according to eq.(12) with $\beta = 2, 3$ and $[2] = [1] - 1$ because of the relabelling difference (87) between the Lagrangian and Eulerian series. Note that the momentum map which is the coefficient of ϕ_t in (90) is exactly the same as the momentum in front of ψ_t in (91). The reason for this goes back to the degeneration of the Eulerian and Lagrangian series into one and the fact that it is the momentum map that is the important element in the general construction (12). In the dispersionless limit the Lagrangians (89), (90), (91) reduce to the gas dynamics Lagrangians (63), (64) and (65) with $\gamma = 2$.

6 Kaup-Broer System

There is another completely integrable dispersive version of the $\gamma = 2$ case of gas dynamics which is the Kaup-Broer system [20], [21]. The triangular invertible differential substitution

$$\rho = \eta + \varepsilon u_x \quad (92)$$

transforms the Kaup-Boussinesq system (79) into the Kaup-Broer system

$$\begin{aligned} u_t &= u u_x + \eta_x + \varepsilon u_{xx} \\ \eta_t &= (\eta u)_x - \varepsilon \eta_{xx} \end{aligned} \quad (93)$$

which also has three local Hamiltonian structures [22]. For the Kaup-Broer system the conserved Hamiltonians in the Eulerian series are given by

$$\mathcal{H}_0^{KBr} = u\eta \quad (94)$$

$$\mathcal{H}_1^{KBr} = \frac{1}{2}[u^2\eta + \eta^2 - 2\varepsilon\eta u_x] \quad (95)$$

$$\mathcal{H}_2^{KBr} = \frac{1}{2}[u^3\eta + 3u\eta^2 + 6\varepsilon\eta u u_x - 4\varepsilon^2 u_x \eta_x], \quad (96)$$

$$\begin{aligned} \mathcal{H}_3^{KBr} = & \frac{1}{4}u^4\eta + \frac{3}{2}u^2\eta^2 + \frac{1}{2}\eta^3 + \varepsilon\left(\frac{3}{2}\eta^2 u_x - u^3\eta_x\right) \\ & + \varepsilon^2(2u^2\eta_{xx} - \eta u_x^2 - \eta_x^2) - 2\varepsilon^3\eta_x u_{xx} \end{aligned} \quad (97)$$

which can be obtained from (83)-(85) through the substitution (92). The first Hamiltonian operator for the Kaup-Broer system is given by (45) and the second Hamiltonian operator

$$J_1^{KBr} = \begin{pmatrix} D & \frac{1}{2}D u + \varepsilon D^2 \\ \frac{1}{2}u D - \varepsilon D^2 & \frac{1}{2}(\eta D + D\eta) \end{pmatrix} \quad (98)$$

can be obtained from (80) of the Kaup-Boussinesq system using the substitution (92).

For Kaup-Broer system we introduce the potentials

$$\eta = w_x, \quad \psi = w + \varepsilon\varphi_x \quad (99)$$

and arrive at the first Lagrangian

$$\mathcal{L}_1^{KBr} = \mathcal{H}_{-1}^{KBq}\varphi_t + \mathcal{H}_{-2}^{KBr}w_t - 2\mathcal{H}_1^{KBr}(w_x, \varphi_x, w_{xx}, \varphi_{xx}, \dots) \quad (100)$$

but now we can derive two further Lagrangians using the recursion operator obtained from the Hamiltonian operators (98) and (45). Following our procedure of section 2 we find the second Lagrangian

$$\mathcal{L}_2^{KBr} = (\mathcal{H}_0^{KBr} - 2\varepsilon w_{xx})\varphi_t + \mathcal{H}_{-1}^{KBr}w_t - 2\mathcal{H}_2^{KBr}(w_x, \varphi_x, w_{xx}, \varphi_{xx}, \dots) \quad (101)$$

which is the same as the Lagrangian of Kaup-Boussinesq system (91) subject to the differential substitution (92). Similarly we find

$$\begin{aligned} \mathcal{L}_3^{KBr} = & (\mathcal{H}_1^{KBr} + 2\varepsilon^2 w_{xxx})\varphi_t \\ & + (\mathcal{H}_0^{KBr} + \varepsilon\varphi_x\varphi_{xx} + \varepsilon^2\varphi_{xxx})w_t - 2\mathcal{H}_3^{KBr}(\varphi_x, w_x, \dots) \end{aligned} \quad (102)$$

as the third Lagrangian for the Kaup-Broer equations (93). As in the case of Kaup-Boussinesq, these Lagrangians reduce to $\gamma = 2$ gas dynamics Lagrangians in the dispersionless limit. In the Kaup-Broer Lagrangians we find another example of the general formula (12) for Lagrangians.

7 Nonlinear Shrödinger equation

We shall consider the nonlinear Shrödinger equation in the 2-component real version

$$\begin{aligned} v_t &= \left[\frac{v^2}{2} + \eta + \varepsilon^2 \left(\frac{\eta_{xx}}{\eta} - \frac{\eta_x^2}{2\eta^2} \right) \right]_x \\ \eta_t &= (\eta v)_x, \end{aligned} \quad (103)$$

which is a reaction-diffusion system. Again this reduces to the $\gamma = 2$ case of gas dynamics in the dispersionless limit. This version of NLS can be obtained by another triangular differential substitution

$$u = v + \varepsilon \eta_x / \eta \quad (104)$$

from the Kaup-Broer system.

NLS has the same first local Hamiltonian structure (45) as in the case of Kaup-Boussinesq or Kaup-Broer systems. Once again the second Hamiltonian operator for NLS can be found by the transformation (104) from the second Hamiltonian operator (98) of the Kaup-Broer system. Thus for the 2-component real version of NLS the second Hamiltonian operator is given by

$$J_2^{NLS} = \begin{pmatrix} D + \varepsilon^2 \left\{ -\frac{1}{2} [(\eta^{-1})_{xx} D + D(\eta^{-1})_{xx}] \right\} & \frac{1}{2} D v \\ \frac{1}{2} v D & \frac{1}{2} (\eta D + D \eta) \end{pmatrix} \quad (105)$$

and the conserved Hamiltonians are

$$\mathcal{H}_{-2}^{NLS} = v \quad (106)$$

$$\mathcal{H}_{-1}^{NLS} = \eta \quad (107)$$

$$\mathcal{H}_0^{NLS} = v\eta \quad (108)$$

$$\mathcal{H}_1^{NLS} = \frac{1}{2} \left(\eta v^2 + \eta^2 - \varepsilon^2 \frac{\eta_x^2}{\eta} \right) \quad (109)$$

$$\mathcal{H}_2^{NLS} = \frac{1}{2} \left[\eta v^2 + 3v\eta^2 + \varepsilon^2 \left(v_x \eta_x - 3 \frac{v \eta_x^2}{\eta} \right) \right] \quad (110)$$

$$\begin{aligned} \mathcal{H}_3^{NLS} &= \frac{3}{4} \eta^2 v^2 + \frac{1}{4} \eta^3 + \frac{1}{8} v^4 \eta + \varepsilon^4 \left(\frac{\eta_{xx}^2}{2\eta} - \frac{5\eta_x^4}{24\eta^3} \right) \\ &\quad + \varepsilon^2 \left(v^2 \eta_{xx} - \frac{5}{4} \eta_x^2 - \frac{3}{4} \frac{\eta_x^2}{\eta} v^2 - \frac{1}{2} v_x^2 \eta \right) \\ &\quad \dots \end{aligned} \quad (111)$$

which forms an infinite sequence combining both Eulerian and Lagrangian series according to (87).

In order to construct the Lagrangians for NLS we introduce the potentials

$$\begin{aligned} v &= z_x \\ z &= \varphi - \varepsilon \ln w_x \end{aligned} \quad (112)$$

and the first Lagrangian

$$\mathcal{L}_1^{NLS} = \mathcal{H}_{-1}^{NLS} z_t + \mathcal{H}_{-2}^{NLS} w_t - 2\mathcal{H}_1^{NLS}(w_x, z_x, \dots) \quad (113)$$

is the classical result. Once again we shall use the techniques of section 2 to construct higher Lagrangians with the recursion operator obtained from (105) and (45). We obtain two higher Lagrangians for NLS

$$\mathcal{L}_2^{NLS} = \mathcal{H}_0^{NLS} z_t + \left[\mathcal{H}_{-1}^{NLS} + \varepsilon^2 \left(\frac{w_{xxx}}{w_x} - \frac{w_{xx}^2}{w_x^2} \right) \right] w_t - 2\mathcal{H}_2^{NLS}(w_x, z_x, \dots) \quad (114)$$

and

$$\begin{aligned} \mathcal{L}_3^{NLS} &= \left\{ \mathcal{H}_0^{NLS} + \varepsilon^2 \left[z_{xxx} + \left(\frac{z_x w_{xx}}{w_x} \right)_x \right] \right\} w_t \\ &\quad + \left(\mathcal{H}_1^{NLS} + 2\varepsilon^2 w_{xxx} \right) z_t - 2\mathcal{H}_3^{NLS}(z_x, w_x, \dots). \end{aligned} \quad (115)$$

that are local functionals of the potentials. Here again, in the dispersionless limit we find the $\gamma = 2$ gas dynamics Lagrangians. The remarkable strength of the general expression (12) for new Lagrangians is manifest.

8 Boussinesq Equation

In order to discuss the bi-Hamiltonian structure and the Lagrangians for the Boussinesq equation in a unified framework we first turn to its dispersionless limit. For polytropic gas dynamics we had

$$\rho_t = (\rho u)_x, \quad u_t = \left(\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1} \right)_x \quad (116)$$

with its first nontrivial commuting flow

$$\rho_y = u_x, \quad u_y = \left(\frac{\rho^{\gamma-2}}{\gamma-2} \right)_x \quad (117)$$

both of which reduce to a second order quasi-linear wave equation [13]. If we express Boussinesq equation in the form

$$\rho_{yy} - \left(\frac{1}{2} \rho^2 - \varepsilon^2 \rho_{xx} \right)_{xx} = 0 \quad (118)$$

or

$$\rho_y = u_x, \quad u_y = \left(\frac{\rho^2}{2} - \varepsilon^2 \rho_{xx} \right)_x \quad (119)$$

as a first order evolutionary system and compare its dispersionless limit to polytropic gas dynamics, we find that it corresponds to the commuting flow for $\gamma = 4$. The completely integrable dispersive equation

$$\begin{aligned} \rho_t &= \left[\rho u - 2\varepsilon^2 u_{xx} \right]_x, \\ u_t &= \left[\frac{u^2}{2} + \frac{1}{3} \rho^3 - \frac{3}{2} \varepsilon^2 (2\rho \rho_{xx} + \rho_x^2) + 2\varepsilon^4 \rho_{xxxx} \right]_x \end{aligned} \quad (120)$$

is the commuting flow to the Boussinesq equation.

This system admits bi-Hamiltonian structure [24] with the Hamiltonian operators (45) and

$$J_2^B = \begin{pmatrix} \rho D + D\rho - 8\varepsilon^2 D^3 & 3u D + 2u_x \\ 3Du - 2u_x & 8(\rho^2 D + D\rho^2) + 8\varepsilon^4 D^5 \\ & -\varepsilon^2 [5(\rho D^3 + D^3 \rho) - 3(\rho_{xx} D + D\rho_{xx})] \end{pmatrix} \quad (121)$$

which are compatible. The conserved Hamiltonian densities for the Boussinesq system are given by

$$\mathcal{H}_{-1}^E = \rho, \quad (122)$$

$$\mathcal{H}_0^E = \rho u, \quad (123)$$

$$\mathcal{H}_1^E = \frac{1}{4} \left[2\rho u^2 + \frac{1}{3} \rho^4 + \varepsilon^2 (6\rho \rho_x^2 + 4u_x^2) + 4\varepsilon^4 \rho_{xx}^2 \right],$$

$$\mathcal{H}_2^E = \frac{1}{28} \left[\frac{14}{3} \rho u^3 + \frac{7}{3} \rho^4 u + 14\varepsilon^2 (2u u_x^2 + 4\rho^2 \rho_x u_x + 3u \rho \rho_x^2) \right] \quad (124)$$

$$+ 28\varepsilon^4 (u \rho_{xx}^2 + \rho_x^2 u_{xx} + 4\rho \rho_{xx} u_{xx}) + 64\varepsilon^6 \rho_{xxx} u_{xxx} \quad (125)$$

in the Eulerian sequence and we have also

$$\mathcal{H}_{-1}^L = u, \quad (126)$$

$$\mathcal{H}_0^L = u^2 + \frac{1}{3}\rho^3 + \varepsilon^2\rho_x^2, \quad (127)$$

$$\mathcal{H}_1^L = \frac{1}{3}u^3 + \frac{1}{3}\rho^3u - \varepsilon^2u(4\rho\rho_{xx} + 3\rho_x^2) + \frac{16}{5}\varepsilon^4u_{xx}\rho_{xx}, \quad (128)$$

$$\begin{aligned} \mathcal{H}_2^L = & \frac{2}{3}u^4 + \frac{4}{3}\rho^3u^2 + \frac{4}{45}\rho^6 + \varepsilon^2\left(\frac{28}{3}\rho^3\rho_x^2 + 4u^2\rho_x^2 + 32\rho u\rho_xu_x + 8\rho^2u_x^2\right) \\ & + \varepsilon^4\left(\frac{136}{5}\rho^2\rho_{xx}^2 - \frac{248}{5}\rho_x^4 + \frac{128}{5}uu_{xx}\rho_{xx} + \frac{16}{5}u_x^2\rho_{xx} + \frac{96}{5}\rho u_{xx}^2\right) \\ & + \varepsilon^6\left(32\rho\rho_{xxx}^2 - \frac{592}{15}\rho_{xx}^3 + \frac{64}{5}u_{xxx}^2\right) + \frac{64}{5}\varepsilon^8\rho_{xxxx}^2 \end{aligned} \quad (129)$$

in the Lagrangian sequence. The Hamiltonian function of Boussinesq system with the first order Hamiltonian operator in Darboux form (45) is $\frac{1}{2}H_0^L$. We note that the system (119) for the Boussinesq equation differs from all dispersive integrable examples we encountered earlier in that its familiar Hamiltonian function (128) is in the Lagrangian sequence. This is because Boussinesq equation is the family of commuting flows to the regular gas dynamics hierarchy. The first commuting higher flow for the Boussinesq system (120) has the Hamiltonian function (124) in the Eulerian series.

By introducing potentials

$$u = \varphi_x, \quad \rho = \psi_x \quad (130)$$

we can obtain two local Lagrangian densities for the Boussinesq system. First we have the classical Lagrangian

$$\mathcal{L}_1^{B(L)} = \mathcal{H}_{-1}^{L\gamma=4}\psi_y + \mathcal{H}_{-1}^{E\gamma=4}\varphi_y - \mathcal{H}_0^{L\gamma=4} \quad (131)$$

$$\mathcal{L}_1^{B(E)} = \mathcal{H}_{-1}^{L\gamma=4}\psi_t + \mathcal{H}_{-1}^{L\gamma=4}\varphi_t - 2\mathcal{H}_1^{E\gamma=4} \quad (132)$$

for Boussinesq system and its first nontrivial commuting flow (120). The second Lagrangians are given by

$$\mathcal{L}_2^{B(L)} = (\mathcal{H}_0^{E\gamma=4} - 4\varepsilon^2\varphi_{xxx})\varphi_y + [\mathcal{H}_0^{L\gamma=4} - 5\varepsilon^2(\psi_x\psi_{xx})_x + 4\varepsilon^4\psi_{xxxx}] \psi_y - \mathcal{H}_1^{L\gamma=4} \quad (133)$$

$$\mathcal{L}_2^{B(E)} = (\mathcal{H}_0^{E\gamma=4} - 4\varepsilon^2\varphi_{xxx})\varphi_t + [\mathcal{H}_0^{L\gamma=4} - 5\varepsilon^2(\psi_x\psi_{xx})_x + 4\varepsilon^4\psi_{xxxx}] \psi_t - 2\mathcal{H}_2^{E\gamma=4} \quad (134)$$

according to the general construction of Lagrangians in (12). Here we see also that the Lagrangian for the commuting flow is obtained by flipping the Hamiltonian functions between the Lagrangian and Eulerian series while keeping the momenta fixed. In section 4 we had constructed Lagrangians

for gas dynamics using the Hamiltonians from the Eulerian series in the potential part of the Lagrangian. The general formula (12) can readily be used to construct Lagrangians for the commuting flow (119) by this simple flip in the potential.

9 Conclusion

This is the first time it has been possible to write a Lagrangian for polytropic gas dynamics that is local in the original field variables, namely the density and velocity. It is a result of the general expression (12) that serves to identify immediately multi-Lagrangians for completely integrable systems. What is even more remarkable is that this is only the first element in an infinite series of such local Lagrangians for polytropic gas dynamics.

It is worth emphasizing again that the scheme we have presented in section 2 is a universal one for the construction of multi-Lagrangians appropriate to evolutionary systems. The expressions (12) and (21) for Lagrangians of completely integrable systems has general validity. We note that (12) with $\alpha = 1$ is true even in the case of non-integrable equations, provided the equations are presented in the form of conservation laws and the system admits one further conserved quantity, namely the Hamiltonian. We have discussed in detail the higher Lagrangians for the completely integrable non-linear evolution equations of polytropic gas dynamics, Kaup-Boussinesq, Kaup-Broer, NLS and Boussinesq equations all of which bear out the universal applicability of (12) in the construction of higher Lagrangians. We have also presented the lower Lagrangians (21) fully for KdV and partially for gas dynamics owing to the difficulty of writing the second modified variables in closed form.

The invariance group of these multi-Lagrangians and their Noether currents should prove to be of interest in discovering new hidden symmetries of fluid mechanics. We did not discuss this important issue here. Recently Jackiw and co-authors [25] have used hidden symmetries in the classical Lagrangian for fluid mechanics to construct very interesting field theory models of fluid mechanics. Multi-Lagrangians may prove to be of interest in this connection also.

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